J. Appl. Maths Mechs Vol. 56, No. 4, pp. 501–507, 1992
Printed in Great Britain.

0021–8928/92 $24.00 + .00

STABILITY OF THE EQUILIBRIUM OF A SYSTEM WITH SINGLE-SIDED CONSTRAINTS AND THE SIGN-DEFINITENESS OF A PENCIL OF QUADRATIC FORMS IN A CONE†

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(*Received* 21 *May* 1991)

The problem of the stability of the equilibrium of a mechanical system constrained by single-sided constraints is considered. It is assumed that in the equilibrium state the constraints are realized, but there are no reactions. It is shown that the conditions for the equilibrium of such systems to be stable can be obtained by an analysis of the sign-definiteness of a pencil of quadratic forms in a cone. A method of solving this algebraic problem is given.

1. CONSIDER a mechanical system with generalized coordinates $q = (q_1, \ldots, q_n)$ and Lagrangian $L = T_2 - \Pi$, where

$$T_2 = T_2(q, q^{\cdot}) = \tfrac{1}{2}\langle q^{\cdot}, A(q)q^{\cdot}\rangle, \qquad \Pi = \Pi(q) \tag{1.1}$$

the matrix-function $A(q)$ and the function $\Pi(q)$ are analytical in some domain D of a space R^n, and angle brackets denote the scalar product. We assume that the motion of the system is restricted by single-sided ideal constraints of the form

$$q_i \geqslant 0, \quad i = 1, 2, \ldots, m, \quad m \leqslant n \tag{1.2}$$

We define the perturbed motion as follows. In the time intervals between momentary impulses the motion of the system is governed by the laws of mechanics for unconstrained systems [1]. The coefficients of restitution at the impulsive instants are equal to unity. If at an impulsive instant not one, but several constraints are realized ("multiple impulse") then the continuation of the motion after the impulse may be non-unique. The possible non-uniqueness in the determination of the perturbed motion is not an obstacle to investigating the stability of the equilibrium, if in the definition of stability one requires smallness of any perturbed motion for small perturbations of the initial conditions.

According to the principle of virtual displacement in a position of equilibrium of system (1.1) (1.2) the inequalities [1, 2]

$$\langle \partial\Pi/\partial q, \delta q\rangle \geqslant 0, \quad \delta q = (\delta q_1, \ldots, \delta q_n) \tag{1.3}$$

are satisfied. Below we shall consider a position of equilibrium $q^* \in D$ in which the constraints on $q_1, \ldots, q_m$ are realized, (i.e. $q_i^* = 0$, $i = 1, \ldots, m$) but the reactions are equal to zero.

In condition (1.3) the independent quantities δq_i take non-negative values for $i = 1, \ldots, m$ and are unrestricted in sign for $i = m+1, \ldots, n$, while the quantities $-\partial\Pi/\partial q_i$ are proportional to the reactions at the constraints. Hence at position q^* the condition

$$\partial\Pi/\partial q = 0 \tag{1.4}$$

is satisfied, i.e. q^* is a stationary point of the function $\Pi(q)$. Condition (1.4) is necessary and sufficient for the equilibrium of the system in position q^* and the absence of reactions at the constraints $q_i \geqslant 0$, $i = 1, \ldots, m$. We have the following generalized Lagrange–Dirichlet theorem [3].

† *Prikl. Mat. Mekh.* Vol. 56, No. 4, pp. 597–603, 1992.

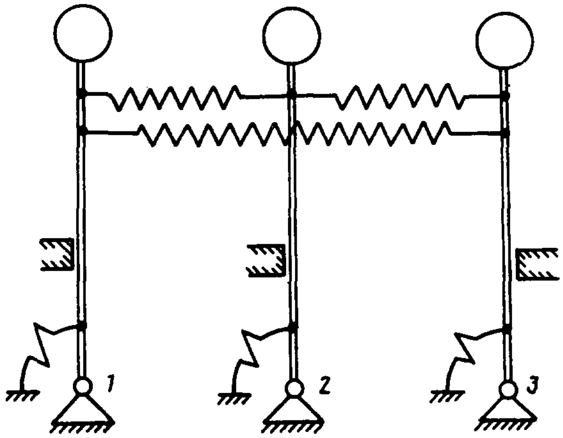

Fig. 1.

Theorem 1. If at the position $q^* = (0, \ldots, 0, q^*_{m+1}, \ldots, q^*_n) \in D$ the function $\Pi(q)$ has a strict local minimum in the domain $q_i \geqslant 0$, $i = 1, \ldots, m$, $q \in D$, then this position has Lyapunov stability.

We also have the following assertion generalizing Lyapunov's theorem [4].

Theorem 2. If the function $\Pi(q)$ does not have a local minimum at the point $q^* = (0, \ldots, 0, q^*_{m+1}, \ldots, q^*_n) \in D$ in the domain $q_i \geqslant 0$, $i = 1, \ldots, m$, $q \in D$, and the absence of the minimum is established by second-order terms in the expansion of $\Pi(q)$ in powers of q in a neighbourhood of q^*, then the position of q^* is unstable.

Below we shall consider systems in which the function $\Pi(q)$ depends not only on the coordinates q, but also on a numerical parameter α taking values in a domain R of the numerical axis, in which the position q^* is an equilibrium for all values of $\alpha \in R$. This means that conditions (1.4) are satisfied at $q = q^*$ for all $\alpha \in R$. Examples of such systems are shown in Figs 1 and 2.

The system in Fig. 1 consists of three rods rotating in a vertical plane. The coordinates q_1, q_2, q_3 are the angles of inclination of the rods from the vertical. The positive direction of q_1 and q_2 is clockwise, and that of q_3 is anticlockwise. The constraints $q_i \geqslant 0$ are implemented by buffers. Helical springs attached to fixed points and to the rods create restoring moments proportional to the q_i, and have coefficients of rigidity κ_i. Springs joining the ends of the rods have coefficients of rigidity k_{ij}. The distance from the points of attachments of the rods to

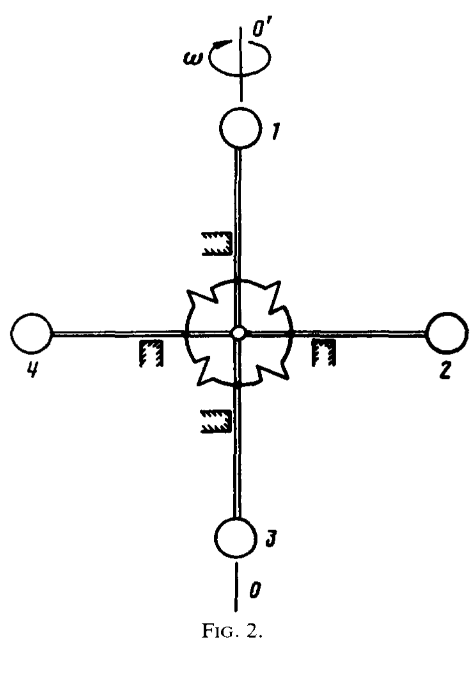

Fig. 2.

the springs are all equal to l. The distances between the points of attachment of rods 1, 2 and 2, 3 horizontally are the same and equal to r. The mass m_0 of each of three identical spheres attached to the ends of the rods plays the role of the parameter α. The potential energy of the system has the form

$$\Pi(q,\alpha) = \frac{1}{2}\sum_{i=1}^{3} \kappa_i q_i^2 + m_0 lg \sum_{i=1}^{3} (1-\cos q_i) + K_{12}^-(r) + K_{23}^+(r) + K_{13}^+(2r)$$

$$K_{ij}^{\pm}(r) = \tfrac{1}{2} k_{ij} l \{[(\cos q_2 - \cos q_1)^2 + (\sin q_2 - \sin q_1 + r/l)^2]^{1/2} - r/l\}^2$$

where g is the acceleration due to gravity. Here $n = m = 3$, $\alpha = m_0$ and $R = [0, +\infty)$. Condition (1.4) at $q^* = (0, 0, 0)$ will be satisfied for all $\alpha \in R$.

The system of Fig. 2 consists of four rods with ends hinged to a common point and rotating in a common plane. The angles of inclination of rods 1 and 3 to the vertical are chosen to be coordinates q_1 and q_3. Coordinates q_2 and q_4 are the angles of inclination of rods 2 and 4 from the horizontal. The positive direction for angles q_1 and q_4 is clockwise, and anticlockwise for angles q_2 and q_3. The constraints $q_i \geqslant 0$ are implemented by buffers. The rods are joined by helical springs which produce moments

$$M_{12} = \kappa_{12}|q_1 + q_2|, \quad M_{23} = \kappa_{23}|q_2 - q_3|$$

$$M_{34} = \kappa_{34}|q_3 + q_4|, \quad M_{14} = \kappa_{14}|q_1 - q_4|$$

(taking account of the chosen positive directions). It is assumed that there is no force of gravity. The plane in which the rods move rotates about the O–O' axis with constant angular velocity ω. Masses m_i are attached to the ends of the rods and l_i is the length of the ith rod.

The variable potential energy of the system has the form

$$\Pi = -\tfrac{1}{2}\omega^2 [m_1(l_1 \sin q_1)^2 + m_3(l_3 \sin q_3)^2 + m_2(l_2 \cos q_2)^2 + m_4(l_4 \cos q_4)^2] + \\ + \tfrac{1}{2}[\kappa_{12}(q_1+q_2)^2 + \kappa_{14}(q_1-q_4)^2 + \kappa_{23}(q_2-q_3)^2 + \kappa_{34}(q_3+q_4)^2] \tag{1.5}$$

Here $\alpha = \omega^2$, $m = n = 4$ and condition (1.4) at $q^* = (0, 0, 0, 0)$ is satisfied for all $\alpha \in R = [0, +\infty)$.

The problem under consideration is to obtain conditions on the parameter α which ensure that the equilibrium position of system (1.1), (1.2) is stable by Theorems 1 and 2.

2. When there are no single-ended constraints the satisfaction of the conditions of the Lagrange–Dirichlet theorem is verified by applying the Sylvester criterion to the matrix of second-order partial derivatives of the potential energy in the postion of equilibrium. The presence of single-sided constraints $q_i \geqslant 0$, $i = 1, \ldots, m$ and the parameter α complicates the problem considerably. We put $x = q - q^*$ and represent the function $\Pi(q, \alpha)$ in the form

$$\Pi(q,\alpha) = \Pi(q^*,\alpha) + \tfrac{1}{2}Q(x,\alpha) + P(x,\alpha) \tag{2.1}$$

for any $\alpha \in R$, where $Q(x, \alpha)$ is a quadratic form in the variables x_i with coefficients depending on α, and $|P(x, \alpha)|/|x|^2 \to 0$ as $|x| \to 0$ for all $\alpha \in R$. We will confine ourselves to the case when $\Pi(q, \alpha)$ depends linearly on α. Then

$$Q(x,\alpha) = (x, Ax) + \alpha(x, Bx) \tag{2.2}$$

is a pencil of quadratic forms, where A and B are symmetric matrices. We will denote by

$$C = \{x \in R^n \mid x_i \geqslant 0, \ i = 1, \ldots, m\} \tag{2.3}$$

a cone in the space R^n.

We have the following assertion [5].

Lemma 1. For a strict local minimum to exist in the domain $C \cap D$ for the function $\Pi(q, \alpha)$ at the point q^* it is necessary for $Q(x, \alpha) \geqslant 0$ for $x \in C$, and sufficient for $Q(x, \alpha) > 0$ for $x \in C$.

We denote by r the set of those values of α for which the quadratic form $Q(x, \alpha)$ (2.2) is positive-definite. A quadratic form which takes positive values in a domain C will be called positive-definite in the domain C, and we denote by $\rho \subseteq R$ the set of those values of α for which the form $Q(x, \alpha)$ is positive-definite in the domain C. A symmetric matrix will be called positive-definite in the domain C if the corresponding quadratic form is positive-definite in that domain.

It is obvious that $r \subseteq \rho$. Both these sets are open. We will formulate and prove an assertion on the structures of the sets r and ρ.

Lemma 2. The set ρ is an open interval of the form (ρ_1, ρ_2). If $B \neq 0$, then the possible cases are: (a) $\rho_1 > -\infty$, $\rho_2 < +\infty$; (b) $\rho_1 = -\infty$, $\rho_2 < +\infty$; and (c) $\rho_1 > -\infty$, $\rho_2 = +\infty$. The same also applies to the set $r = (r_1, r_2)$.

Proof. We shall prove the assertions for the set ρ. The assertions for r are proved similarly. We assume that ρ is not an interval. Then there are numbers $\alpha < \beta < \gamma$ such that $\alpha, \gamma \in \rho$ and $\beta \notin \rho$. Consequently, there is an $x' \in C$ ($x' \in R^n$ for the r case) satisfying the inequality $Q(x', \beta) \leq 0$. Then because $Q(x', \alpha) > 0$, it follows from (2.2) and using $\beta - \alpha > 0$ that $(x', Bx') < 0$, which contradicts the condition $\gamma \in \rho$ because

$$Q(x', \gamma) = Q(x', \beta) + (\gamma - \beta)(x', Bx') < 0$$

If the matrix B is non-zero, then the interval ρ cannot coincide with the entire real axis. The lemma is proved.

We will denote by $\bar{\rho}$ and $\bar{r}$ the closures of the sets ρ and r. If $\alpha \notin \bar{\rho}$ then a vector x' exists such that $Q(x', \alpha) < 0$. From Lemmas 1 and 2 the following theorem follows.

Theorem 3. Suppose the function $\Pi(q, \alpha)$ is analytic in some neighbourhood D of the point $q^* = (0, \ldots, 0, q^*_{m+1}, \ldots, q^*_n) \in D$, that at the point q^* condition (1.4) is satisfied for all $\alpha \in R$, the quadratic form in expansion (2.1) has the form (2.2), and $B \neq 0$. Then one can find an interval $\rho = (\rho_1, \rho_2)$ such that for $\alpha \in \rho$ the function $\Pi(q, \alpha)$ has a strict local minimum in the domain $C \cap D$ at the point q^*, while for $\alpha \notin \rho_0$ there is no local minimum at q^*.

For $\alpha \in \rho \cap R$ Theorem 1 guarantees Lyapunov stability at the position of equilibrium q^*, while for $\alpha \notin \rho \cap R$ Theorem 2 guarantees instability at q^*.

3. We shall describe a method for finding the boundaries ρ_1 and ρ_2 of the interval of positive-definiteness for the pencil of quadratic forms (2.2) in the cone C. We write $N = \{1, 2, \ldots, n\}$, $M = \{1, 2, \ldots, m\}$, which are sets of indices. For any subset $I \subseteq M$ we denote the number of indices in the set I by $m(I)$, and put $n(I) = n - m + m(I)$. For any symmetric $n \times n$ matrix D we denote by $D(I)$ the $n(I) \times n(I)$ matrix obtained from D by eliminating rows and columns with numbers $i \notin I$.

Suppose $I \subseteq M$, $I \neq 0$ and $E_\alpha \equiv A + \alpha B$. We denote by $C(I)$ a cone of the form $x \in R^{n(1)} | x_i \geq 0$, $i \in I$ and by $\rho(I) = [\rho_1(I), \rho_2(I)]$ the interval of positive-definiteness of the pencil of quadratic forms with matrix $E_\alpha(I)$ in the cone $C(I)$. The notation $r(I)$, $r_1(I)$ and $r_2(I)$ has similar meanings.

The proposed method of finding the interval $\rho = \rho(M)$ is based on inductively reducing the dimensionality and uses a result of [6]. The first step of the induction is given by Lemma 3.

Lemma 3. Let $i \in M$. For the matrix $E_\alpha(\{i\})$ to be positive-definite in the cone $C(\{i\})$ it is necessary and sufficient for it to be positive-definite throughout the space $R^{n(\{i\})}$, i.e. $\rho(\{i\}) = r(\{i\})$.

Proof. We only need to prove necessity. We assume that the matrix $E_\alpha(\{i\})$, positive-definite in the cone $C(\{i\})$, is not positive-definite in $R^{n(\{i\})}$. Then one can find a vector $y \in R^{n(\{i\})}$, $y \in C(\{i\})$ such that $\langle y, E_\alpha(\{i\}) y \rangle \leq 0$. This means that $y_i < 0$. But then $z = -y \in C(\{i\})$, which contradicts the assumption, because $\langle z, E_\alpha(i) z \rangle = \langle y, E_\alpha(i) y \rangle$.

An $n(I) \times n(I)$ matrix $D(I)$ will be called minimal with respect to the index set I if it is not positive-definite, but all matrices $D(I \backslash i)$, $i \in I$ are positive-definite. Using the above notation the minimality of the matrix $E_\alpha(I)$ with respect to the set I is expressed by the satisfaction of the conditions $\alpha \notin r(I)$ and $\alpha \in r(I \backslash \{i\})$ for all $i \in I$. If for a given α the matrix E_α is not positive-definite and the matrices $E_\alpha(\{i\})$ are positive-definite for all $i \in I$ (and this is necessary for the positive definiteness of the matrix E_α in the domain C because of Lemma 3), then we have found sets I with respect to which the matrices $E_\alpha(I)$ are minimal.

The assertion below is given without the proof (the proof almost literally repeats that of Theorem 1 from [6]).

Theorem 4. Suppose $I \subseteq M$ and $m(I) \geqslant 2$. Then to satisfy the conditions $\alpha \in \rho(I)$ it is necessary and sufficient to satisfy one of three conditions: (a) $\alpha \in r(I)$, (b) $\alpha \notin r(I)$, for all $i \in I$ $\alpha \in \rho(I \backslash \{i\})$ and there is at least one $i_0 \notin I$ such that $\alpha \in r(I \backslash \{i_0\})$ (i.e. the matrix $E_\alpha(I)$ is not minimal with respect to I), (c) the matrix $E_\alpha(I)$ is minimal with respect to I and, for some (arbitrary) vector $y \neq 0$ satisfying the conditions $Q(y, \alpha) \leqslant 0$, $y_i = 0$ for $i \notin I$, has amongst its components y_i, $i \in I$ components of opposite sign.

Remark. If the matrix $E_\alpha(I)$ is minimal with respect to I, then by definition it is not positive-definite and a vector y which gives the quadratic form $Q(x, \alpha)$ a non-negative value and satisfying the conditions $y_i = 0$ for $i \in I$ can always be found. If for some α the matrix $E_\alpha(I)$ is minimal with respect to I, then at least one of the conditions

$$r_1(I) > \max_{i \in I} r_1(I \backslash \{i\}) \tag{3.1}$$

$$r_2(I) < \min_{i \in I} r_2(I \backslash \{i\}) \tag{3.2}$$

is satisfied.

Suppose, to fix our ideas, that condition (3.1) is satisfied. From the definition of $r_1(I)$ the matrix $E_\alpha(I)$ is positive-definite for $\alpha = r_1(I) + \epsilon$ for sufficiently small $\epsilon > 0$ and is not positive-definite for $\alpha < r_1(I)$. This means that for $r_1(I) \neq -\infty$ one can find a vector $y \neq 0$ satisfying the conditions $Q[y, r_1(I)] = 0$ and $y_i = 0$ for $i \notin I$. This vector is the solution of the system of linear equations

$$\sum_{j \in N \backslash M \cup I} (a_{ij} + \alpha b_{ij}) x_j = 0, \quad i \in N \backslash M \cup I$$
$$x_i = 0, \quad i \in M \backslash I \tag{3.3}$$

for $\alpha = r_1(I)$, where a_{ij} and b_{ij} are elements of the matrices A and B. Because for sufficiently small $\epsilon > 0$ we have $Q(y, r_1(I) + \epsilon) > 0$, then $(y, By) < 0$ and, consequently, $Q(y, \alpha) < 0$ for all $\alpha < r_1(I)$. Thus, if condition (3.1) is satisfied, and the vector $y \neq 0$ satisfies condition (3.3), then the vector y can be used to verify the condition in Theorem 4 for any $r_1(I) \leqslant \alpha < \max_{i \in I} r_1(I \backslash \{i\})$. The presence in the vector y of components of opposite sign means that $y \neq 0$ and $y \notin C \cup (-C)$, where $-C = \{z \mid -z \in C\}$.

If (3.2) is satisfied, then replacing α by $r_2(I)$ in (3.3), we obtain a vector $y \neq 0$ satisfying the conditions $y_i = 0$ for $i \in I$ and $[Q(y, r_2(I)] = 0$ and $Q(y, \alpha) < 0$ for $\max_{i \in I} r_2(I \backslash \{i\}) < \alpha \leqslant r_2(I)$.

If $r_1(I) > -\infty$, then Eq. (3.3) has non-trivial solutions for $\alpha = r_1(I)$. We denote any one of them by $y_1(I)$. If $r_2(I) < +\infty$ we similarly use $y_2(I)$ to denote an arbitrary non-trivial solution of system (3.3) by $\alpha = r_1(I)$. For those $I \subseteq M$ which contain at least two indices we put

$$r_1'(I) = \max_{i \in I} r_1(I \backslash \{i\}), \quad \rho_1'(I) = \max_{i \in I} \rho_1(I \backslash \{i\})$$

From Theorem 4 and its remark there follows a recurrence relation for the sequential definition of the values of $\rho_k(I)$, $k = 1, 2$, $I \subseteq M$.

For any $i = 1, \ldots, m$ we have $\rho_1(\{i\}) = r_1(\{i\})$. Suppose $m(I) \geqslant 2$. Then if $r_1(I) = \rho_1'(I)$, then $\rho_1(I) = \rho_1'(I)$. If however $r_1(I) > \rho_1'(I)$, then

$$\rho_1(I) = \begin{cases} r_1(I), \text{ if } r_1'(I) = \rho_1'(I) > -\infty \text{ and } y_1(I) \in C \cup (-C) \\ \rho_1'(I) \text{ otherwise} \end{cases} \tag{3.4}$$

For any $i = 1, \ldots, m$ we similarly have $\rho_2(\{i\}) = r_1(\{i\})$. Suppose $m(I) \geqslant 2$. If $r_2(I) = \rho_2'(I)$, then $\rho_2(I) = \rho_2'(I)$. If however $r_2(I) < \rho_2'(I)$, then

$$\rho_2(I) = \begin{cases} r_2(I), \text{ if } r_2'(I) = \rho_2'(I) < +\infty \text{ and } y_2(I) \in C \cup (-C) \\ \rho_2'(I) \text{ otherwise} \end{cases} \tag{3.5}$$

$$r_2' = \min_{i \in I} r_2(I \backslash \{i\}), \quad \rho_2' = \min_{i \in I} \rho_2(I \backslash \{i\})$$

4. The proposed method enables us to determine inductively the values $\rho_1(M)$, $\rho_2(M)$ of the boundaries of the interval of positive definiteness of the pencil of quadratic forms (2.2) in the cone C assuming that $r_1(I)$, $r_2(I)$ are known for all $I \subseteq M$. We shall consider the method for determining $r_1(I)$ and $r_2(I)$. Suppose $I \subseteq M$, $m(I) < m$ and $r_1(I)$, $r_2(I)$ are known. If $\alpha \in [r_1(I), r_2(I)]$, then the symmetric $n(I) \times n(I)$ matrices $E_\alpha(I)$ are positive-definite. Then according to Sylvester's criterion

TABLE 1

I	$r(I)$	$y_1(I)$	$y_2(I)$	$\rho(I)$
{1}	$(-\infty; 2)$	—	—	$(-\infty; 2)$
{2}	$(-3; \infty)$	—	—	$(-3; \infty)$
{3}	$(-\infty; 1.5)$	—	—	$(-\infty; 1.5)$
{4}	$(-3; \infty)$	—	—	$(-3; \infty)$
{1, 2}	$(-2.86; 1.86)$	$[1; -7.29]$	$[1; -0.21]$	$(-3; 2)$
{1, 3}	$(-\infty; 1.5)$	—	—	$(-\infty; 1.5)$
{1, 4}	$(-2.39; 1.39)$	$[1; 3.28]$	$[1; 0.46]$	$(-2.39; 1.39)$
{2, 3}	$(-2.5; 1)$	$[1; 0.25]$	$[1; 2]$	$(-2.5; 1)$
{2, 4}	$(-3; \infty)$	—	—	$(-3; \infty)$
{3, 4}	$(-2.89; 1.39)$	$[1; -9.0]$	$[1; -0.21]$	$(-3; 1.5)$
{1, 2, 3}	$(-2.32; 0.89)$	$[-0.6; 3.8; 1]$	$[-0.37; 0.61; 1]$	$(-2.5; 1)$
{1, 2, 4}	$(-2.21; 1.21)$	$[0.4; -0.5; 1]$	$[2.1; -0.5; 1]$	$(-2.39; 1.39)$
{1, 3, 4}	$(-2.23; 1.08)$	$[0.3; -0.13; 1]$	$[0.72; -1.19; 1]$	$(-2.39; 1.39)$
{2, 3, 4}	$(-2.35; 0.85)$	$[3.1; 1; -1.53]$	$[0.52; 1; -0.26]$	$(-2.5; 1)$
{1, 2, 3, 4}	$(-1.27; 0)$	$[0.6; -0.95; -0.53; 1]$	$[1; -1; -1; 1]$	$(-2.39; 1)$

for any $J = I \cup \{i\}$, $i \in M \backslash I$ the matrix $E_\alpha(J)$ is positive-definite if and only if the matrix $E_\alpha(I)$ is positive-definite and the condition det $E_\alpha(I) > 0$ is satisfied. Because by Lemma 2 the sets $r(I)$ and $r(J)$ are intervals and $r(J) \subset r(I)$, the interval $r(J)$ is extracted from $r(I)$ by one or two roots of the equation $\det E_\alpha(J) = 0$ lying in the interval $r(I)$, and the condition $\det E_\alpha(J) > 0$ for $\alpha \in r(J)$.

If $m = n$, then for any $i \in M$ the matrices $E_\alpha(\{i\})$ are numbers and the interval $r(\{i\})$ is given by the condition $a_{ii} + \alpha b_{ii} > 0$. Here either $r_1(\{i\}) = -\infty$ or $r_2(\{i\}) = +\infty$. All the remaining $r(I)$ are given by induction, as was shown above. If however $m < n$, then the interval $r(\phi)$ is also given by induction with successive borderings, starting with an arbitrary index $i \in N \backslash M$, as a result of the considerations given above.

5. As an example we consider the system of Fig. 2. It is required to find an interval for the rotational frequency ω for which Theorems 1 and 2 guarantee stability for the equilibrium state $q_1^* = q_2^* = q_3^* = q_4^* = 0$. Putting $\alpha = \omega^2$ in formula (1.5), we obtain expressions for the matrices A and B in the expansion (2.1), (2.2).

$$A = \begin{Vmatrix} \kappa_{12} + \kappa_{14} & \kappa_{12} & 0 & -\kappa_{14} \\ \kappa_{12} & \kappa_{12} + \kappa_{23} & -\kappa_{23} & 0 \\ 0 & -\kappa_{23} & \kappa_{23} + \kappa_{34} & \kappa_{34} \\ -\kappa_{14} & 0 & \kappa_{34} & \kappa_{14} + \kappa_{34} \end{Vmatrix}$$

$$B = \operatorname{diag}\{-m_1 l_1^2, m_2 l_2^2, -m_3 l_3^2, m_4 l_4^2\}$$

Suppose $\kappa_{12} = \kappa_{34} = 1$, $\kappa_{23} = \kappa_{14} = 2$, $l_1 = l_2 = l_3 = l_4 = 1$, $m_1 = 1.5$, $m_2 = m_4 = 1$, $m_3 = 2$. The interval $r(I)$ is determined by the method given in Sec. 4. The interval $\rho(I)$ is established by relations (3.4) and (3.5). The results of the calculations are given in Table 1, the third and fourth columns of which contain the vectors $y_1(I)$ and $y_2(I)$. For $I = \{1\}$, $\{2\}$, $\{3\}$ and $\{4\}$ these vectors are undefined because $m(I) = 2$. For the subsets $I = \{1, 3\}$ and $\{2, 4\}$ $y_1(I)$ and $y_2(I)$ are undefined because the conditions $r_k(I) = \rho_k'(I)$, $k = 1, 2$ are satisfied. Thus $\rho = \rho(M) = (-2.5, 1)$. Because α stands for ω^2, $R = [0, +\infty)$. Theorem 1 ensures stability at the equilibrium position q^* for $0 \leq \omega < 1$ and Theorem 2 ensures instability at q^* when $\omega > 1$.

I wish to thank Ye. S. Pyatnitskii for his interest.

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Translated by R.L.Z.

J. Appl. Maths Mechs Vol. 56, No. 4, pp. 507–517, 1992
Printed in Great Britain.

0021–8928/92 $24.00 + .00

CONSEQUENCES OF NON-INTEGRABLE PERTURBATION OF INTEGRABLE CONSTRAINTS: NON-LINEAR EFFECTS OF MOTION NEAR THE EQUILIBRIUM MANIFOLD†

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(*Received* 15 *January* 1990)

A general analysis of non-linear oscillations of conservative non-holonomic systems is presented: the choice of special coordinates in a neighbourhood of the equilibrium manifold, the analytic structure of normal forms of higher approximations beginning with the second, the use of the energy integral, and the explicit form of approximate solutions.

THE EQUATIONS of motion of such systems to a first approximation have been considered both for the case of the critical point of the potential energy [1] and for the case of an arbitrary regular equilibrium [2]. The approximated constraint equations were integrated in the first paper, but not in the second. This gave rise to essentially irrelevant polemics, because in the general case it is proper to consider the neighbourhood of a manifold of equilibria rather than an isolated equilibrium [3].

Numerous investigations of the stability of non-holonomic systems (see the review [4]) have been largely based on the first approximation equation, mainly for the non-conservative case. Theorems on instability at the critical point are exceptions: the method of Chetayev functions was used [4] and asymptotic motions were constructed [5].

If all eigenvalues lie on the imaginary axis, the difference between the exact solution and the first approximation remains small only for finite times. It follows that interesting qualitative effects at long times in the motion of conservative non-holonomic systems about an equilibrium can only be found by turning to higher approximations, i.e. by utilizing the method of normal forms (see, e.g. [6, 7]). The first such investigation was the paper by Markeyev [8].

Below it is shown that non-linear oscillations of systems with non-integrable constraints can be naturally considered in the framework of the general concept of weak non-holonomicity [9], and their normal forms possess definite characteristic features.

† *Prikl. Mat. Mekh.* Vol. 56, No. 4, pp. 604–614, 1992.